Fundamental Length,Deformed Density Matrix and New View on the Black Hole Information Paradox

A.E.Shalyt-Margolin *

National Center of Particles and High Energy Physics, Bogdanovich Str. 153, Minsk 220040, Belarus

PACS: 03.65; 05.20
Keywords: fundamental length,density matrix,deformed density matrix

Abstract

In this paper Quantum Mechanics with Fundamental Length is chosen as Quantum Mechanics at Planck's scale. This is possible due to the presence in the theory of General Uncertainty Relations (GUR). Here Quantum Mechanics with Fundamental Length is obtained as a deformation of Quantum Mechanics. The distinguishing feature of the proposed approach in comparison with previous ones, lies on the fact that here density matrix subjects to deformation whereas so far commutators have been deformed. The density matrix obtained by deformation of quantum-mechanical density one is named throughout this paper density pro-matrix, which at low energy limit turns to the density matrix. This transition corresponds to non-unitary one from Quantum Mechanics with GUR to Quantum mechanics. Below the implications of obtained results are enumerated.New view on the Black Holes Information Paradox are discussed

*Fax (+375) 172 326075; e-mail: a.shalyt@mail.ru;alexm@hep.by

1 Introduction

In this paper Quantum Mechanics with Fundamental Length is chosen as Quantum Mechanics at Planck's scale. This is possible due to the presence in the theory of General Uncertainty Relations. Here Quantum Mechanics with Fundamental Length is obtained as a deformation of Quantum Mechanics. The distinguishing feature of the proposed approach in comparison with previous ones, lies on the fact that here density matrix subjects to deformation whereas so far commutators have been deformed. The density matrix obtained by deformation of quantum-mechanical density one is named throughout this paper density pro-matrix. Within our approach two main features of Quantum Mechanics are conserved: the probabilistic interpretation of the theory and the well-known measuring procedure corresponding to that interpretation.It was shown also, inflationary model contains two different (unitary non-equivalent) Quantum Mechanics: the first one describes nature at the Planck scale and it is based on the GUR. The second one is obtained as a limit transition from Planck scale to low energy one and it is based on the Heisenberg uncertainty relations. The interpretation of obtained results as well as their implications are discussed below, in particular the new view on the Black Holes Information Paradox.

2 General Uncertainty Relations and Fundamental Length

Let's start considering the Heisenberg Uncertainty relation (position-momentum) [1] :

$$\Delta x \geq \frac{\hbar}{\Delta p}. \tag{1}$$

In the last 14-15 years a lot of papers were issued in which authors using string theory [2], gravitation [3], Quantum theory of black holes [4] and other methods [5] shown that Heisenberg Uncertainty relations should be modified. In particular, a high energy addition have to appear

$$\triangle x \geq \frac{\hbar}{\triangle p} + \alpha L_p^2 \frac{\triangle p}{\hbar}. \tag{2}$$

Where L_p - the Planck length, $L_p = \sqrt{\frac{G\hbar}{c^3}} \simeq 1,6\ 10^{-35} m$ and $\alpha > 0$ is a constant. In paper [3] was shown this constant can be chosen equal to 1.

However, here we will use α as an arbitrary constant without any concrete value. The inequality (2) is quadratic with respect to $\triangle p$

$$\alpha L_p^2(\triangle p)^2 - \hbar \triangle x \triangle p + \hbar^2 \leq 0 \tag{3}$$

and from it follows the fundamental length is

$$\triangle x_{min} = 2\sqrt{\alpha} L_p \tag{4}$$

Since further we are going to base only on the existence of fundamental length it is necessary to point out this fact was established not only from GUR. For instance, in [6],[7] using an ideal experiment dealing with gravitation field it was obtained the lower bound on limit length, which was improved in [8] without GUR to an estimate of the type $\sim L_p$. In what follows we will use the abbreviations GUR for General Uncertainty Relations(2) and UR for Heisenberg Uncertainty Relations (2) correspondingly.

3 Density matrix and its deformation

Let's consider in some detail the equation (4). Squaring it left and right side, we obtain

$$(\overline{\Delta \widehat{X}^2}) \geq 4\alpha L_p^2 \tag{5}$$

or in terms of density matrix

$$Sp[(\rho \widehat{X}^2) - Sp^2(\rho \widehat{X})] \geq 4\alpha L_p^2 > 0 \tag{6}$$

where $\widehat{X}$ is the coordinate operator. Expression (6) gives the measuring rule used in QM. However, in the case considered here, in comparison with QM, the right part of (6) cannot be done arbitrarily near to zero since it is limited by $l_{min}^2 > 0$, where due to GUR $l_{min} \sim L_p$.

Apparently, this may be due to the fact that QMFL with GUR (2) is unitary non-equivalent to QM with UR. Actually, in QM the left-hand side of (6) can be chosen arbitrary closed to zero, whereas in QMFL this is impossible. But if two theories are unitary equivalent then, the form of their spurs should be retained. Besides, a more important aspect is contributing to unitary non-equivalence of these two theories: QMFL contains three fundamental constants (independent parameters) G, c and $\hbar$, whereas QM contains only one: $\hbar$. Within an inflationary model (see [10]),

QM is the low-energy limit of QMFL (QMFL turns to QM) for the expansion of the Universe. In this case, the second term in the right-hand side of (2) vanishes and GUR turn to UR. A natural way for studying QMFL is to consider this theory as a deformation of QM, turning to QM at the low energy limit (during the expansion of the Universe after the Big Bang). We will consider precisely this option. However differing from authors of papers [4], [5] and others, we do not deform commutators, but density matrix, leaving at the same time the fundamental quantum-mechanical measuring rule (6) without changes. Here the following question may be formulated: how should be deformed density matrix conserving quantum-mechanical measuring rules in order to obtain self-consistent measuring procedure in QMFL? For answering to the question we will use the R-procedure. For starting let us to consider R-procedure both at the Planck's energy scale and at the low-energy one. At the Planck's scale $a \approx il_{min}$ or $a \sim iL_p$, where i is a small quantity. Further a tends to infinity and we obtain for density matrix

$$Sp[\rho a^2] - Sp[\rho a]Sp[\rho a] \simeq l_{min}^2 \;\; or \;\; Sp[\rho] - Sp^2[\rho] \simeq l_{min}^2/a^2.$$

Therefore:

1. When $a < \infty$, $Sp[\rho] = Sp[\rho(a)]$ and $Sp[\rho] - Sp^2[\rho] > 0$. Then, $Sp[\rho] < 1$ that corresponds to the QMFL case.

2. When $a = \infty$, $Sp[\rho]$ does not depend on a and $Sp[\rho] - Sp^2[\rho] \to 0$. Then, $Sp[\rho] = 1$ that corresponds to the QM case.

How should be points 1 and 2 interpreted? How does analysis above-given agree to the main result from [18] [1]? It is in full agreement. Indeed, when state-vector reduction (R-procedure) takes place in QM then, always an eigenstate (value) is chosen exactly. In other words, the probability is equal to 1. As it was pointed out in the above-mentioned point 1 the situation changes when we consider QMFL: it is impossible to measure coordinates exactly since it never will be absolutely reliable. We obtain in all cases a probability less than 1 ($Sp[\rho] = p < 1$). In other words, any R-procedure in QMFL leads to an eigenvalue, but only with a probability less than 1. This probability is as near to 1 as far the difference between measuring

[1]"... there cannot be any physical state which is a position eigenstate since a eigenstate would of course have zero uncertainty in position"

scale a and l_{min} is growing, or in other words, when the second term in (2) becomes insignificant and we turn to QM. Here there is not a contradiction with [18]. In QMFL there are not exact coordinate eigenstates (values) as well as there are not pure states. In this paper we do not consider operator properties in QMFL as it was done in [18] but density-matrix properties.

The properties of density matrix in QMFL and QM have to be different. The only reasoning in this case may be as follows: QMFL must differ from QM, but in such a way that in the low-energy limit a density matrix in QMFL must coincide with the density matrix in QM. That is to say, QMFL is a deformation of QM and the parameter of deformation depends on the measuring scale. This means that in QMFL $\rho = \rho(x)$, where x is the scale, and for $x \to \infty$ $\rho(x) \to \widehat{\rho}$, where $\widehat{\rho}$ is the density matrix in QM.

Since on the Planck's scale $Sp[\rho] < 1$, then for such scales $\rho = \rho(x)$, where x is the scale, is not a density matrix as it is generally defined in QM. On Planck's scale we name $\rho(x)$ "density pro-matrix". As follows from the above, the density matrix $\widehat{\rho}$ appears in the limit

$$\lim_{x \to \infty} \rho(x) \to \widehat{\rho}, \tag{7}$$

when GUR (2) turn to UR and QMFL turns to QM.

Thus, on Planck's scale the density matrix is inadequate to obtain all information about the mean values of operators. A "deformed" density matrix (or pro-matrix) $\rho(x)$ with $Sp[\rho] < 1$ has to be introduced because a missing part of information $1 - Sp[\rho]$ is encoded in the quantity l_{min}^2/a^2, whose specific weight decreases as the scale a expressed in units of l_{min} is going up.

4 QMFL as a deformation of QM by density matrix

Here we are going to describe QMFL as a deformation of QM using the density pro-matrix formalism. In this context density pro-matrix has to be understood as a deformed density matrix in QMFL. As fundamental deformation parameter we will use $\beta = l_{min}^2/x^2$, where x is the scale.

Definition 1.

Any system in QMFL is described by a density pro-matrix $\rho(\beta) = \sum_i \omega_i(\beta)|i><i|$, where

1. $0 < \beta \leq 1/4$;

2. The vectors $|i>$ form a full orthonormal system;

3. $\omega_i(\beta) \geq 0$ and for all i there is a finite limit $\lim\limits_{\beta\to 0} \omega_i(\beta) = \omega_i$;

4. $Sp[\rho(\beta)] = \sum_i \omega_i(\beta) < 1$, $\sum_i \omega_i = 1$;

5. For any operator B and any β there is a mean operator B, which depends on β:

$$< B >_\beta = \sum_i \omega_i(\beta) < i|B|i > .$$

At last, in order to match our definition with the result of section 2 the next condition has to be fulfilled:

$$Sp[\rho(\beta)] - Sp^2[\rho(\beta)] \approx \beta, \tag{8}$$

from which we can find the meaning of the quantity $Sp[\rho(\beta)]$, which satisfies the condition of definition:

$$Sp[\rho(\beta)] \approx \frac{1}{2} + \sqrt{\frac{1}{4} - \beta}. \tag{9}$$

From point 5. it follows, that $< 1 >_\beta = Sp[\rho(\beta)]$. Therefore for any scalar quantity f we have $< f >_\beta = fSp[\rho(\beta)]$. In particular, the mean value $< [x_\mu, p_\nu] >_\beta$ is equal to

$$< [x_\mu, p_\nu] >_\beta = i\hbar\delta_{\mu,\nu} Sp[\rho(\beta)] \tag{10}$$

We will call density matrix the limit $\lim\limits_{\beta\to 0} \rho(\beta) = \rho$. It is evident, that in the limit $\beta \to 0$ we turn to QM. Here we would like to verify, that two cases described above correspond to the meanings of β. In the first case when β is near to 1/4. In the second one when it is near to zero.
From the definitions given above it follows that $< (j >< j) >_\beta = \omega_j(\beta)$. From which the condition of completeness on β is
$< (\sum_i |i >< i|) >_\beta = < 1 >_\beta = Sp[\rho(\beta)]$. The norm of any vector $|\psi >$, assigned to β can be defined as
$< \psi|\psi >_\beta = < \psi|(\sum_i |i >< i|)_\beta|\psi > = < \psi|(1)_\beta|\psi > = < \psi|\psi > Sp[\rho(\beta)]$, where $< \psi|\psi >$ is the norm in QM, or in other words when $\beta \to 0$. By

analogy, for probabilistic interpretation the same situation takes place in the described theory, but only changing ρ by $\rho(\beta)$.

Some remarks:

I. The considered above limit covers at the same time Quantum and Classical Mechanics. Indeed, since $\beta = l_{min}^2/x^2 = G\hbar/c^3x^2$, so we obtain:

   a. $(\hbar \neq 0, x \rightarrow \infty) \Rightarrow (\beta \rightarrow 0)$ for QM;
   b. $(\hbar \rightarrow 0, x \rightarrow \infty) \Rightarrow (\beta \rightarrow 0)$ for Classical Mechanics;

II. The parameter of deformation β should take the meaning $0 < \beta \leq 1$. However, as we can see from (9), and as it was indicated in the section 2, $Sp[\rho(\beta)]$ is well defined only for $0 < \beta \leq 1/4$.That is if $x = il_{min}$ and $i \geq 2$ then, there is not any problem. At the very point with fundamental length $x = l_{min} \sim L_p$ there is a singularity, which is connected with the appearance of the complex value of $Sp[\rho(\beta)]$, or in other words it is connected with the impossibility of obtain a diagonalized density pro-matrix at this point over the field of real numbers. For this reason definition 1 at the initial point do not has any sense.

III. We have to consider the question about solutions (8). For instance, one of the solutions (8), at least at first order on β is $\rho^*(\beta) = \sum_i \alpha_i exp(-\beta)|i><i|$, where all $\alpha_i > 0$ do not depend on β and their sum is equal to 1, that is $Sp[\rho^*(\beta)] = exp(-\beta)$. Indeed, we can easy verify that

$$Sp[\rho^*(\beta)] - Sp^2[\rho^*(\beta)] = \beta + O(\beta^2). \qquad (11)$$

Note that in the momentum representation $\beta = p^2/p_{max}^2$, where $p_{max} \sim p_{pl}$ and p_{pl} is the Planck's momentum. When present in matrix elements, $exp(-\beta)$ can damp the contribution of great momenta in a perturbation theory.

IV. It is clear, that in the proposed description of states, which have a probability equal to 1, or in others words pure states can appear only in the limit $\beta \rightarrow 0$, or when all states $\omega_i(\beta)$ except one of them are equal to zero, or when they tend to zero at this limit.

V. We suppose, that all definitions concerning density matrix can be transferred to the described above deformation of Quantum Mechanics (QMFL) changing the density matrix ρ by the density pro-matrix $\rho(\beta)$ and turning then to the low energy limit $\beta \to 0$. In particular, for statistical entropy we have

$$S_\beta = -Sp[\rho(\beta)\ln(\rho(\beta))]. \tag{12}$$

The quantity S_β, evidently never is equal to zero, since $\ln(\rho(\beta)) \neq 0$ and, therefore S_β may be equal to zero only at the limit $\beta \to 0$.

5 Some Implications

1. If we carry out a measurement in a defined scale,we cannot consider a density pro-matrix, density pro-matrix with a precision, which exceed some limit of order $\sim 10^{-66+2n}$, where 10^{-n} is the scale in which the measurement is carried out. In most of the known cases this precision is quite enough for considering density pro-matrix the density matrix. However, at the Planck scale, where Quantum Gravity effects cannot be neglected and energy is of the Planck order the difference between $\rho(x)$ and $\widehat{\rho}$ have to be considered.

2. At the Planck scale the notion of wave function of the Universe, introduced by J.A. Wheeler and B. deWitt [9] does not work and in this case quantum gravitation effects can be described only with the help of density pro-matrix ρ.

3. Since density pro-matrix ρ depends on the scale in which the measurement is carried out, so the evolution of the Universe within inflation model paradigm [10] is not an unitary process, because, otherwise the probability p_i would be conserved.

4. As density pro-matrix ρ for a pure state does not exist, so $\ln\rho \neq 0$ and statistical entropy $S = -Sp[\rho\ln\rho] \neq 0$ is never equal to zero at the Planck scale, and condition $S = -Sp[\rho\ln\rho] = 0$ can be used only with a certain degree of precision depending on the scale.

6 On the problem of black holes information paradox

The obtained above results give us a new approach to the solution of Hawking problem on coherence (unitary) and information loss in black holes [11]. Indeed, the relation (7) describes the limit transition from density pro-matrix ρ to density matrix $\widehat{\rho}$ or in others words, from GUR to UR. Is it possible the inverse transition from density matrix $\widehat{\rho}$ to density pro-matrix ρ and correspondingly, from (1) to (2) ? The answer is affirmative. This transition is possible when matter are absorbed by a black hole if we consider,that quantum gravity effects are important when we are trying to describe physical effects in a black hole, as it was do in (2)[12]. Thereby we have the next symmetric and equivalent transitions:

I.GUR(Big Bang,Origin Singularity)$\mapsto$ UR $\mapsto$ GUR(Black Holes,Singularities);

II.Density Pro-Matrix(mix states)$\mapsto$ Density Matrix(mix states,pure states)$\mapsto$ Density Pro-Matrix(mix states).

In all papers on coherence and information loss in black holes (for instance, [13],[14]) so far, the authors have handled with the right side I and II or another words, with the transitions UR $\mapsto$ GUR(Black Holes,Singularities), Density Matrix(mix states,pure states)$\mapsto$ Density Pro-Matrix(mix states), which are non-unitary according to the obtained above results. However it is evidently, that it is more rightful if we study I and II completely, in other words, if we add the left sides GUR(Big Bang,Origin Singularity)$\mapsto$ UR Density Pro-Matrix(mix states)$\mapsto$ Density Matrix(mix states,pure states) correspondingly. Then, starting from Density Pro-Matrix(mix states) we come back to Density Pro-Matrix(mix states) and unitarity and information can be restored. It is necessary to remark,that for primordial black holes in I and II their middle part vanishes, since all processes take place in the early universe, and we obtain the same result

Density Pro-Matrix(mix states))$\mapsto$ Density Pro-Matrix(mix states)

Besides that, it is evident, that the appearance of a space-time singularity, except of the initial singularity in Classical Theory, in Quantum Theory means the transition from UR to GUR or from (1) to (2). In the case of initial singularity we have Quantum Theory with GUR from the very

beginning.

7 Conclusion

As it was noted in [15] all known approaches to justify Quantum Gravity one way or another lead to the notion of fundamental length. Besides that GUR (2), which as well lead to that notion are well described within the inflation model [16]. Therefore to understand physics at the Planck scale without these notions, apparently is not possible. Besides that, it is necessary to consider one more aspect of this problem. As it was noted in [17] , when a new physical theory is created, it implies the introduction of a new parameter and the deformation of precedent theory by this parameter. All these deformation parameters are in their essence fundamental constants: G,c and $\hbar$ (more exactly in [17] c $\hbar$ instead of c, $1/c$ is used). It is possible to join some these parameters in an unique theory. For example G and c in the General Theory of the relativity, c and $\hbar$ in QFT. In [17] the next question was formulated : what could be the theory , which contains all three fundamental constants, or in other words all three deformation parameters? From all these follows the question in [17] can be revised: is the theory with fundamental length, the theory which by definition contains all three fundamental parameters $L_p = \sqrt{\frac{G\hbar}{c^3}}$ by definition ?. In [17] the limit transition from one Quantum Mechanics to another, described as $L_p^2/x^2 \rightarrow 0$ can be understood as $L_p \rightarrow 0$, that in the considered case corresponds either $G \rightarrow 0$,$c \rightarrow \infty$ (in the case of Quantum Mechanics), or $G \rightarrow 0$,c tend to a finite quantity (in the case of Relativistic Quantum theory).

References

[1] W.Heisenberg,Uber den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik, Zeitsch.fur Phys,43(1927)172

[2] G.Veneziano,A stringly nature needs just two constant Europhys.Lett.2(1986)199;D.Amati,M.Ciafaloni and G.Veneziano,Can spacetime be probed below the string size? Phys.Lett.B216(1989)41; E.Witten, Reflections on the Fate of Spacetime Phys.Today,49(1996)24

[3] R.J.Adler and D.I.Santiago,On Gravity and the Uncertainty Principle, Mod.Phys.Lett.A14(1999)1371[gr-qc/9904026]

[4] M.Maggiore, A Generalized Uncertainty Principle in Quantum Gravity Phys.Lett.B304(1993)65,[hep-th/9301067]

[5] D.V.Ahluwalia,Wave-Particle duality at the Planck scale: Freezing of neutrino oscillations Phys.Lett. A275 (2000)31, [gr-qc/0002005];Interface of Gravitational and Quantum Realms Mod.Phys.Lett. A17(2002)1135,[gr-qc/0205121]; M.Maggiore,Quantum Groups,Gravity and Generalized Uncertainty Principle Phys.Rev.D49(1994)5182,[hep-th/9305163]; The algebraic structure of the generalized uncertainty principle Phys.Lett.B319(1993)83,[hep-th/9309034];S.Capozziello,G.Lambiase and G.Scarpetta, The Generalized Uncertainty Principle from Quantum Geometry [gr-qc/9910017]

[6] Y.J.Ng, H.van Dam,Measuring the Foaminess of Space-Time with Gravity-Wave Interferometers,Found.Phys.30(2000)795, [gr-qc/9906003]

[7] Y.J.Ng, H.van Dam, On Wigner's clock and the detectability space-time foam with gravitational-wave interferometers Phys.Lett.B477(2000)429,[gr-qc/9911054]

[8] J.C.Baez, S.J.Olson,Uncertainty in Measurment of Distance,[gr-qc/0201030]

[9] J.A.Wheeler,in Battele Rencontres,ed. by C.DeWitt and J.A. Wheeler (Benjamen,NY,1968)123; B.DeWitt,Quantum Thery Gravity I.The Canonical Theory, Phys.Rev.160(1967)1113.

[10] A.H.Guth,Inflation and EternaL Inflation,[astro-ph/0002156]

[11] S.Hawking,Breakdown of Predictability in Gravitational Collapse, Phys.Rev.D14(1976)2460

[12] R.Adler,P.Chen and D.Santiago,The Generalised Uncertainty Principle and Black Hole Remnants,Gen.Rel.Grav.33(2001)2101,[gr-qc/0106080]; P.Chen and R.Adler, Black Hole Remnants and Dark Matter,[gr-qc/0205106]

[13] S.Giddings,The Black Hole Information Paradox,[hep-th/9508151]

[14] A.Strominger, Les Houches Lectures on Black Holes, [hep-th/9501071]

[15] L.Garay,Quantum Gravity and Minimum Length Int.J.Mod.Phys.A.v.A10(1995)145

[16] S.F.Hassan and M.S.Martin, Trans-Plancian Effects in Inflationary Cosmology and Modified Uncertainty Principle, [hep-th/0204110]

[17] L.Faddeev, Mathematical View on Evolution of Physics, Priroda 5(1989)11

[18] A.Kempf,G.Mangano,R.B.Mann,Hilbert Space Representation of the Minimal Length Uncertainty Relation,Phys.Rev.D52(1995)1108[hep-th/9412167]